Tuesday, September 29, 2015

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Problem 5

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by $y = x^2, \;\; y = x^5$ about the x-axis.

Solution. The inner radius is $R_1(x) = x^5$ and the outer radius is $R_2(x) = x^2$. The extremities are $x = 0$ and $x = 1$. So the volume is

$$\begin{aligned} V &= \int_0^1 \pi \left((x^2)^2 - (x^5)^2 \right) \, dx \\ &= \pi \int_0^1 \left(x^4 - x^{10} \right) \, dx \\ &= \pi \left[\frac{1}{5} x^5 - \frac{1}{11} x^{11} \right]_0^1 \\ &= \pi \left(\frac{1}{5} - \frac{1}{11} \right) \\ &= \frac{6\pi}{55}. \end{aligned}$$

Problem 6

Problem. Set up and evaluate the integral that gives the volume of the solid formed by revolving the region bounded by $y = 2, \;\; y = 4 - \frac{x^2}{4}$ about the x-axis.

Solution. The inner radius is $R_1(x) = 2$ and the outer radius is $R_2(x) = 4 - \frac{x^2}{4}$. The extremities are $x = -3$ and $x = 3$. So the volume is

$$
\begin{aligned}
V &= \int_{-3}^{3} \pi \left(\left(4 - \frac{x^2}{4}\right)^2 - 2^2 \right) \, dx \\
&= \pi \int_{-3}^{3} \left(\left(16 - 2x^2 + \frac{x^4}{16}\right) - 4 \right) \, dx \\
&= \pi \int_{-3}^{3} \left(12 - 2x^2 + \frac{x^4}{16} \right) \, dx \\
&= \pi \left[12x - \frac{2}{3}x^3 + \frac{1}{80}x^5 \right]_{-3}^{3} \\
&= \pi \left[\left(36 - \frac{2}{3} \cdot 3^3 + \frac{1}{80} \cdot 3^5 \right) - \left(12x - \frac{2}{3}(-3)^3 + \frac{1}{80}(-3)^5 \right) \right] \\
&= \pi \left[\left(18 + \frac{243}{80} \right) - \left(-18 - \frac{243}{80} \right) \right] \\
&= \pi \left(36 + \frac{486}{80} \right) \\
&= \frac{1683\pi}{40}.
\end{aligned}
$$

Problem 11(b)

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$
\begin{aligned}
y &= \sqrt{x}, \\
y &= 0, \\
x &= 3
\end{aligned}
$$

about the y-axis.

Solution. Because we are revolving about the y-axis, we should write our functions as functions of y. So we have $x = y^2$. Now, the inner radius is $R_1(y) = y^2$ and the outer radius is $R_2(y) = 3$. The extremities are $y = 0$ and $y = \sqrt{3}$. So the volume is

$$\begin{aligned} V &= \int_0^{\sqrt{3}} \pi \left(3^2 - (y^2)^2\right)\ dy \\ &= \pi \int_0^{\sqrt{3}} \left(9 - y^4\right)\ dy \\ &= \pi \left[9y - \frac{1}{5}y^5\right]_0^{\sqrt{3}} \\ &= \pi \left(9\sqrt{3} - \frac{1}{5} \cdot 9\sqrt{3}\right) \\ &= \frac{36\pi}{5}. \end{aligned}$$

Problem 11(d)

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$\begin{aligned} y &= \sqrt{x}, \\ y &= 0, \\ x &= 3 \end{aligned}$$

about the line $x = 6$.

Solution. The inner radius is from $x = 6$ to $x = 3$, so $R_1(y) = 6 - 3 = 3$ and the outer radius is $R_2(y) = 6 - y^2$. The extremities are $y = 0$ and $y = \sqrt{3}$. The volume is

$$\begin{aligned} V &= \int_0^{\sqrt{3}} \pi \left((6 - y^2)^2 - 3^2\right)\ dy \\ &= \pi \int_0^{\sqrt{3}} \left(27 - 12y^2 + y^4\right)\ dy \\ &= \pi \left[27y - 4y^3 + \frac{1}{5}y^5\right]_0^{\sqrt{3}} \\ &= \pi \left(27\sqrt{3} - 4 \cdot (\sqrt{3})^3 + \frac{1}{5}(\sqrt{3})^5\right) \\ &= \frac{84\pi}{5}. \end{aligned}$$

Problem 15

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = x,$$
$$y = 3,$$
$$x = 0$$

about the line $y = 4$.

Solution. We are revolving about a horizontal line, so we must integrate in the x direction and make vertical slices.

The inner radius goes from $y = 4$ to $y = 3$, so $R_1 = 1$. The outer radius goes from $y = 4$ to $y = x$, so $R_2 = 4 - x$. The extremities are from $x = 0$ to $x = 3$. The volume is

$$\begin{aligned}
V &= \int_0^3 \pi\left((4-x)^2 - 1^2\right)\,dx \\
&= \pi \int_0^3 \left(15 - 8x + x^2\right)\,dx \\
&= \pi\left[15x - 4x^2 + \frac{1}{3}x^3\right]_0^3 \\
&= \pi(45 - 36 + 9) \\
&= 18\pi.
\end{aligned}$$

Problem 20

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = 3 - x,$$
$$y = 0,$$
$$y = 2,$$
$$x = 0$$

about the line $x = 5$.

Solution. We are revolving about a vertical line, so we must integrate in the y direction and make horizontal slices.

The inner radius goes from $x = 5$ to $x = 3 - y$, so $R_1 = 5 - (3 - y) = y + 2$. The outer radius goes from $x = 5$ to $x = 0$, so $R_2 = 5$. The extremities are from $y = 0$ to $y = 2$. The volume is

$$\begin{aligned} V &= \int_0^2 \pi \left(5^2 - (y+2)^2\right)\, dy \\ &= \pi \int_0^2 \left(21 - 4y - y^2\right)\, dy \\ &= \pi \left[21y - 2y^2 - \frac{1}{3}y^3\right]_0^2 \\ &= \pi \left(42 - 8 - \frac{8}{3}\right) \\ &= \frac{94\pi}{3}. \end{aligned}$$

Problem 21

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$\begin{aligned} x &= y^2, \\ x &= 4 \end{aligned}$$

about the line $x = 5$.

Solution. We are revolving about a vertical line, so we must integrate in the y direction and make horizontal slices.

The inner radius goes from $x = 5$ to $x = 4$, so $R_1 = 1$. The outer radius goes from $x = 5$ to $x = y^2$, so $R_2 = 5 - y^2$. The extremities are from $y = 0$ to $y = \sqrt{4} = 2$. The volume is

$$\begin{aligned} V &= \int_0^2 \pi \left((5-y^2)^2 - 1\right)\, dy \\ &= \pi \int_0^2 \left(24 - 10y^2 + y^4\right)\, dy \\ &= \pi \left[24y - \frac{10}{3}y^3 + \frac{1}{5}y^5\right]_0^2 \\ &= \pi \left(48 - \frac{80}{3} + \frac{32}{5}\right) \\ &= \frac{832\pi}{15}. \end{aligned}$$

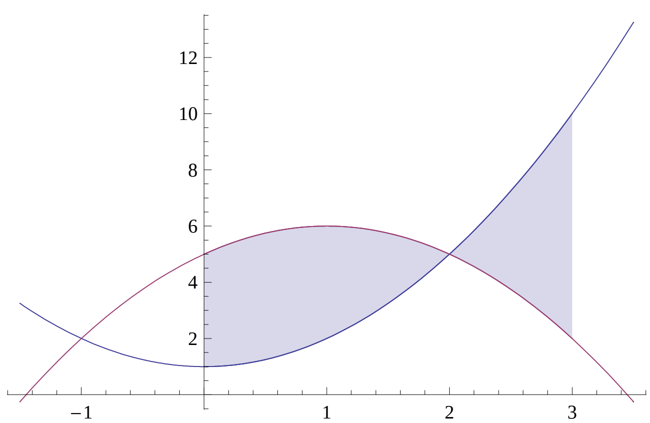
Problem 29

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$\begin{aligned} &y = x^2 + 1, \\ &y = -x^2 + 2x + 5, \\ &x = 0, \\ &x = 3 \end{aligned}$$

about the x-axis

Solution. The graph is



We are revolving about a horizontal line, so we must integrate in the x direction and make vertical slices. Furthermore, we see from the graph that the upper and lower function switch at $x = 2$, so we must set up two integrals.

In the first integral, the inner radius goes from $y = 0$ to $y = x^2+1$, so $R_1 = x^2+1$. The outer radius goes from $y = 0$ to $y = -x^2 + 2x + 5$, so $R_2 = -x^2 + 2x + 5$. The extremities are from $x = 0$ to $x = 2$. The first volume is

$$\begin{aligned} V_1 &= \int_0^2 \pi \left((-x^2 + 2x + 5)^2 - (x^2 + 1)^2\right) \, dx \\ &= \pi \int_0^2 \left(-4x^3 - 8x^2 + 20x + 24\right) \, dx \\ &= \pi \left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 \\ &= \pi \left(-16 - \frac{64}{3} + 40 + 48\right) \\ &= \frac{152\pi}{3}. \end{aligned}$$

In the second integral, the inner radius goes from $y = 0$ to $y = -x^2 + 2x + 5$, so $R_1 = -x^2 + 2x + 5$. The outer radius goes from $y = 0$ to $y = x^2 + 1$, so $R_2 = x^2 + 1$. The extremities are from $x = 2$ to $x = 3$. The second volume is

$$\begin{aligned} V_2 &= \int_2^3 \pi \left((x^2 + 1)^2 - (-x^2 + 2x + 5)^2\right) \, dx \\ &= \pi \int_2^3 \left(4x^3 + 8x^2 - 20x - 24\right) \, dx \\ &= \pi \left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3 \\ &= \pi \left((81 + 72 - 90 - 72) - \left(16 + \frac{64}{3} - 40 - 48\right)\right) \\ &= \pi \left(-9 + \frac{152}{3}\right) \\ &= \frac{125\pi}{3}. \end{aligned}$$

Thus, the combined volume is

$$\begin{aligned} V &= V_1 + V_2 \\ &= \frac{152\pi}{3} + \frac{125\pi}{3} \\ &= \frac{277\pi}{3}. \end{aligned}$$

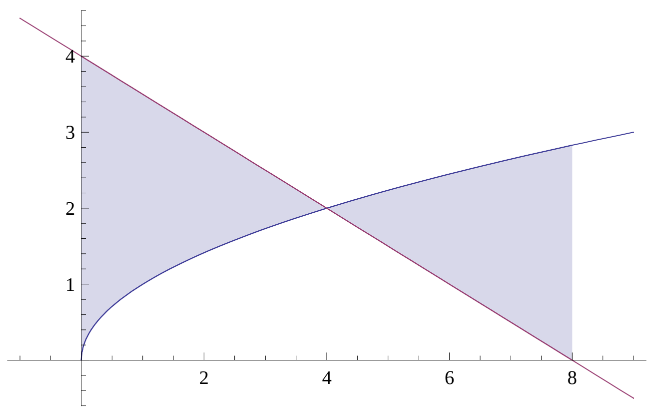
Problem 30

Problem. Find the volume of the solid generated by revolving the region bounded by the graphs of the equations

$$y = \sqrt{x},$$
$$y = -\frac{1}{2}x + 4,$$
$$x = 0,$$
$$x = 8$$

about the x-axis

Solution. The graph is



We are revolving about a horizontal line, so we must integrate in the x direction and make vertical slices. Furthermore, we see from the graph that the upper and lower function switch at $x = 4$, so we must set up two integrals.

In the first integral, the inner radius goes from $y = 0$ to $y = \sqrt{x}$, so $R_1 = \sqrt{x}$. The outer radius goes from $y = 0$ to $y = -\frac{1}{2}x + 4$, so $R_2 = -\frac{1}{2}x + 4$. The extremities are from $x = 0$ to $x = 4$. The first volume is

$$\begin{aligned} V_1 &= \int_0^4 \pi \left((-\frac{1}{2}x + 4)^2 - (\sqrt{x})^2 \right) \, dx \\ &= \pi \int_0^4 \left(\frac{1}{4}x^2 - 5x + 16 \right) \, dx \\ &= \pi \left[\frac{1}{12}x^3 - \frac{5}{2}x^2 + 16x \right]_0^4 \\ &= \pi \left(\frac{16}{3} - 40 + 64 \right) \\ &= \frac{88\pi}{3} \end{aligned}$$

In the second integral, the inner radius goes from $y = 0$ to $y = -x^2 + 2x + 5$, so $R_1 = -x^2 + 2x + 5$. The outer radius goes from $y = 0$ to $y = x^2 + 1$, so $R_2 = x^2 + 1$. The extremities are from $x = 4$ to $x = 8$. The second volume is

$$\begin{aligned} V_2 &= \int_4^8 \pi \left((\sqrt{x})^2 - (-\frac{1}{2}x + 4)^2 \right) \, dx \\ &= \pi \int_4^8 \left(-\frac{1}{4}x^2 + 5x - 16 \right) \, dx \\ &= \pi \left[-\frac{1}{12}x^3 + \frac{5}{2}x^2 - 16x \right]_4^8 \\ &= \pi \left(\left(-\frac{128}{3} + 160 - 128 \right) - \left(-\frac{16}{3} + 40 - 64 \right) \right) \\ &= \pi \left(-\frac{32}{3} + \frac{88}{3} \right) \\ &= \frac{56\pi}{3}. \end{aligned}$$

Thus, the combined volume is

$$\begin{aligned} V &= V_1 + V_2 \\ &= \frac{88\pi}{3} + \frac{56\pi}{3} \\ &= \frac{144\pi}{3} \\ &= 48\pi. \end{aligned}$$

Problem 41

Problem. Find the volume of the solid generated by rotating region R_1 (see diagram) about the line $x = 0$.

Solution. The inner radius is $R_1 = 1 - x$ and the outer radius is $R_2 = 1$. The extremities are $x = 0$ and $x = 1$. The volume is

$$\begin{aligned} V &= \int_0^1 \pi \left(1 - (1-x)^2\right)\, dx \\ &= \pi \int_0^1 \left(2x - x^2\right)\, dx \\ &= \pi \left[x^2 - \frac{1}{3}x^3\right]_0^1 \\ &= \pi \left(1 - \frac{1}{3}\right) \\ &= \frac{2\pi}{3}. \end{aligned}$$

Problem 45

Problem. Find the volume of the solid generated by rotating region R_3 (see diagram) about the line $x = 0$.

Solution. The inner radius is $R_1 = 0$ and the outer radius is $R_2 = x^2$. The extremities are $x = 0$ and $x = 1$. Because $R_1 = 0$, this is the disk method. The volume is

$$\begin{aligned} V &= \int_0^1 \pi \left((x^2)^2\right)\, dx \\ &= \pi \int_0^1 x^4\, dx \\ &= \pi \left[\frac{1}{5}x^5\right]_0^1 \\ &= \frac{\pi}{5}. \end{aligned}$$

Problem 47

Problem. Find the volume of the solid generated by rotating region R_2 (see diagram) about the line $x = 0$.

Solution. Having worked problems 41 and 45, a quick way to get this volume is to subtract the other two volumes from the volume of all three regions rotated about the x-axis. That solid is a cylinder of radius 1 and height 1, so its volume is π. Then the volume of region R_2 rotated must be

$$\begin{aligned} V &= \pi - \frac{2\pi}{3} - \frac{\pi}{5} \\ &= \frac{2\pi}{15}. \end{aligned}$$

Let's use the washer method and see whether we get the same answer. The inner radius is $R_1 = x^2$ and the outer radius is $R_2 = x$. The extremities are $x = 0$ and $x = 1$. The volume is

$$\begin{aligned} V &= \int_0^1 \pi \left(x^2 - (x^2)^2 \right) \, dx \\ &= \pi \int_0^1 \left(x^2 - x^4 \right) \, dx \\ &= \pi \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^1 \\ &= \frac{2\pi}{15}. \end{aligned}$$

Yep.